Lesson

4-5 The Sine and Cosine Functions

Vocabulary

sine function

cosine function

BIG IDEA The values of cos θ and sin θ determine functions with equations $y = \sin x$ and $y = \cos x$ whose domain is the set of all real numbers.

From the exact values of sines, cosines, and tangents you calculated in Lesson 4-4, you can see the shape of a function called the *sine function*.

Mental Math

If gasoline costs $4.00 a gallon and a car gets 25 miles to the gallon, what does it cost for gas per mile?

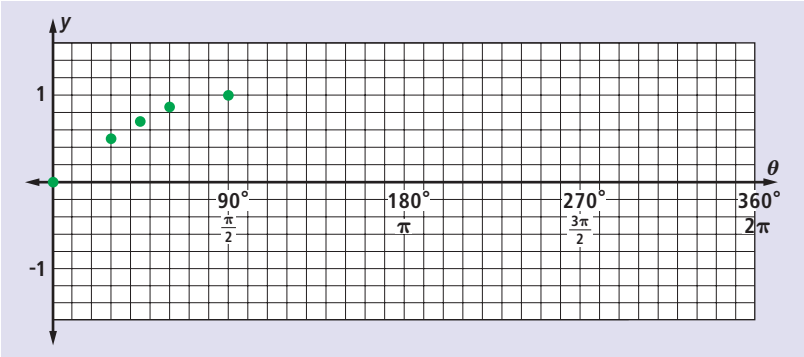
Activity 1

Step 1 The table below contains some exact values of sin θ. It also shows decimal approximations to those values. Complete the table, using a unit circle to help you.

θ (degrees)	0°	30°	45°	60°	90°	120°	135°	150°	180°
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
sin θ (exact)	0	$\frac{1}{2}$	?	?	?	$\frac{\sqrt{3}}{2}$	?	?	?
sin θ (approx.)	0	0.5	?	?	?	?	0.707	?	?

θ (degrees)	210°	225°	240°	270°	300°	315°	330°	360°
θ (radians)	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	?	?	?	2π
sin θ (exact)	?	$-\frac{\sqrt{2}}{2}$	?	?	?	?	?	?
sin θ (approx.)	?	?	?	?	–0.866	?	?	?

Step 2 Here is a graph of the first five points in the first part of the table. Copy this graph, and on it plot the points you found in Step 1. Then draw a smooth curve through the points.



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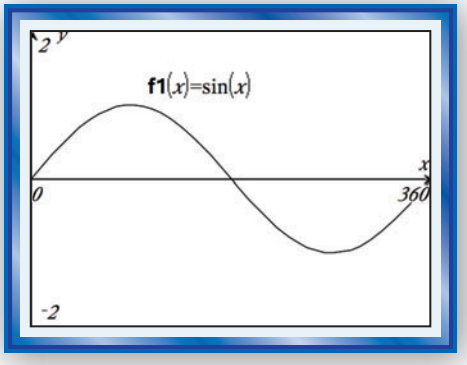
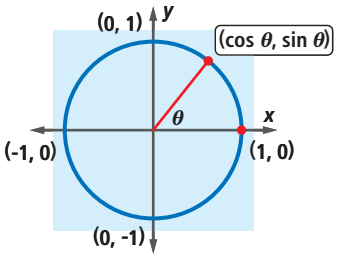
Step 3 Check Step 2 by using a graphing utility to plot $y = \sin\theta$ for $0° \le \theta \le 360°$ and for $0 \le \theta \le 2\pi$.



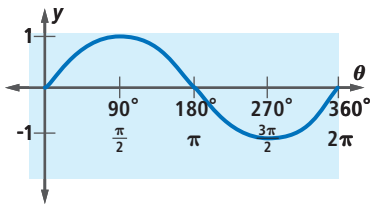
The Graph of the Sine Function



The function that maps each real number θ to the y-coordinate of the image of (1, 0) under a rotation of θ is called the **sine function**. From the unit circle, you can tell that $\sin\theta$ is positive when $0° < \theta < 180°$ and negative when $180° < \theta < 360°$. The maximum value is 1, when $\theta = 90°$, and the minimum value is -1, when $\theta = 270°$.

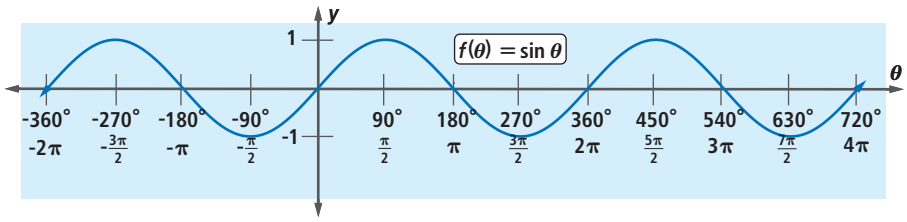
QY

Restate the preceding paragraph for θ in radians.



A graph of the sine function, for $0° \le \theta \le 360°$, is shown at the right. To make it easier to locate zeros, maxima, and minima, the scale on the horizontal axis is in multiples of $\frac{\pi}{2}$ and 90°.

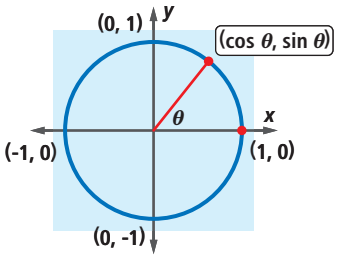
This is one *cycle* of the graph of the sine function. Because the image of (1, 0) under a rotation of θ repeats itself every 2π radians, the y-coordinates in the ordered pairs of the function f with equation $f(\theta) = \sin\theta$ repeat every 2π. Thus, the graph above can be easily extended both to the right and left without calculating any new sine values. The graph of the entire sine function has infinitely many cycles. A graph showing three complete cycles of the sine function appears below.



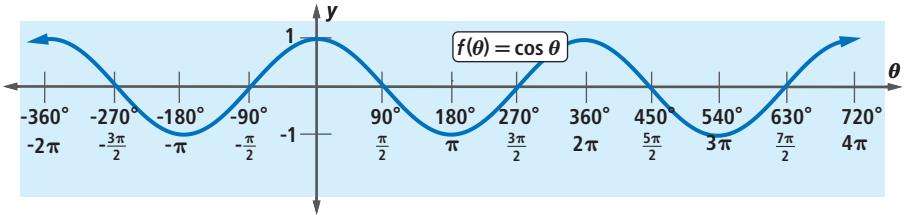
Notice from the graph that the y-intercept of the sine function is 0. The sine function's x-intercepts (zeros) are ..., -2π, $-\pi$, 0, π, 2π, 3π, 4π, ..., that is, the integer multiples of π.

As the graph on the previous page makes clear, the domain of the sine function is the set of real numbers. Because the maximum and minimum values of the sine function are 1 and –1 (the y-intercepts of the unit circle) the range is the interval $-1 \le y \le 1$. Also notice that the graph of the sine function is point-symmetric about the origin. Thus, the sine function is an odd function. This is because of the Opposites Theorem that states for all θ, $\sin(-\theta) = -\sin\theta$.

The Graph of the Cosine Function



Remember that the image of (1, 0) under a rotation of magnitude θ is $(\cos\theta, \sin\theta)$. The function that maps each real number θ to the *first* coordinate of the image of (1, 0) under a rotation of θ is called the **cosine function**. The cosine function has many characteristics like those of the sine function. A graph of the cosine function is shown below.



Activity 2

Use the definitions and graphs of the sine and cosine functions to fill in the table.

	sine function (degrees)	sine function (radians)	cosine function (degrees)	cosine function (radians)
Domain	?	?	?	?
Range	?	?	?	?
Zeros	?	?	?	?
Maxima	$\sin\theta = 1$ when $\theta = 90°, 450°, 810°, \ldots$	?	?	?
Minima	?	?	?	?

Questions

COVERING THE IDEAS

1. a. Identify the domain and the range of the sine function.
   b. Find five values of x such that $\sin x = 0$.
2. a. Sketch a graph of $y = \sin x$ for $0 \le x \le 2\pi$.
   b. Find all values of x on this interval such that $\sin x = 1$.
   c. Find all values of x on this interval for which $\sin x = 0.5$.

3. a. Copy the table below. Fill in exact and approximate values (rounded to three decimal places) for some of the coordinates of points on the graph of the cosine function.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$
$\cos x$ (exact)	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	?	?	$-\frac{\sqrt{3}}{2}$
$\cos x$ (approx.)	1	0.866	0.707	0.5	0	?	?	–0.866

x	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\cos x$ (exact)	–1	?	?	?	?	?	?	?	?
$\cos x$ (approx.)	–1	?	?	?	?	?	?	?	?

b. Use the points from the table to graph $y = \cos x$.

4. a. Sketch a graph of $y = \cos \theta$ for $-360° \leq \theta \leq 720°$.

b. Find five values of θ on this interval for which $\cos \theta = 0$.

5. Describe three ways in which the graph of $y = \cos \theta$ is like the graph of $y = \sin \theta$ and two ways in which the graphs are different.

APPLYING THE MATHEMATICS

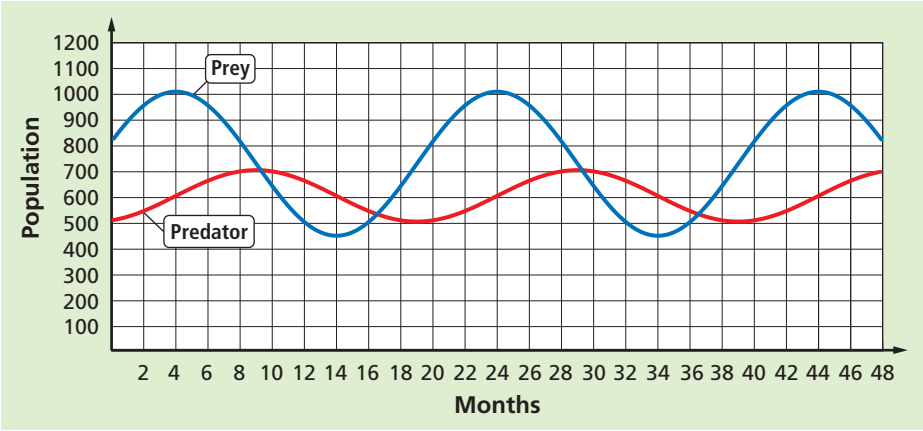
6. Describe the translation with the smallest positive magnitude that maps the graph of $g(x) = \cos x$ onto that of $y = \sin x$.

7. The graph of the sine function is reflection-symmetric over the line with equation $x = \frac{\pi}{2}$.

a. What property of sines is a result of this symmetry?

b. Name two other lines of symmetry for the graph.

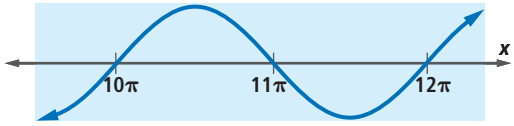
8. In a stable environment, predator-prey populations can be modeled by sine waves. Refer to the graph below.



a. Describe what is happening with the prey population when the predator population is at its peak.

b. Describe what is happening with the prey population when the predators are the fewest.

9. Use the graph of $y = f(x)$ at the right. Suppose f is known to be either the cosine function or the sine function.



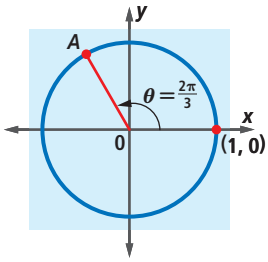
   a. Evaluate $f\left(\frac{23\pi}{2}\right)$.
   b. For what value of x, in the interval from 10π to 12π, does $f(x) = 1$?
   c. Tell whether f is the cosine function or sine function. Justify your answer.

10. The graph of the cosine function is reflection-symmetric to the y-axis. What property of cosines is a result of this symmetry?

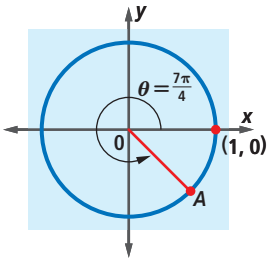
REVIEW

In 11 and 12, *A* is a point on a circle with center at the origin. Find the coordinates of *A* for the given value of θ. (Lesson 4-4)

11.



12.



13. In radians, what is the sum of the measures of the angles of a pentagon? **(Lesson 4-1)**

14. An old 78 RPM record revolves through 78 revolutions in a minute. How many radians is this per second? **(Lesson 4-1)**

15. The measure of an angle is k radians. Convert this measure to degrees. **(Lesson 4-1)**

16. The students in Ms. T. Chare's 1st period geometry class measured their heights h in centimeters and recorded the following five-number summary of their data: $\bar{h} = 165$; min $= 137$; $Q_1 = 154$; median $= 168$; $Q_3 = 174$; max $= 188$. Are there any outliers in the data set? Explain your answer. **(Lesson 1-4)**

EXPLORATION

17. At what angle to the x-axis does the graph of $y = \sin x$ pass through $(0, 0)$? Give numerical and visual evidence supporting your answer.

QY ANSWER

From the unit circle, you can tell that $\sin\theta$ is positive when $0 < \theta < \pi$ and negative when $\pi < \theta < 2\pi$. The maximum value is 1, when $\theta = \frac{\pi}{2}$, and the minimum value is –1, when $\theta = \frac{3\pi}{2}$.